14.3 Partial Derivatives

Goal: Learn how to get the slope in two different directions on a surface.

Entry Task: Consider

$$f(x,y) = x^2y + 5x^3 + y^2$$

Find

(a) $\frac{d}{dx}[f(x,2)] = \frac{d}{dx}[x^2(2) + 5x^3 + (2)^2]$

(b) $\frac{d}{dx}[f(x,3)] = \frac{d}{dx}[x^2(3) + 5x^3 + (3)^2]$

(c) $\frac{d}{dx}[f(x,c)] = \frac{d}{dx}[x^2(c) + 5x^3 + (c)^2]$

Recall the key definition from calculus

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Today we define:

$$\frac{\partial z}{\partial x} = f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial z}{\partial y} = f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h) - f(x,y)}{h}$$

Examples:

$$f(x, y) = x^3 y + x^5 e^{xy^2} + \ln(y)$$

$$g(x, y) = \cos(x^3 + y^4)$$

An Important Note on Variables

To be successful with partial derivatives, you need to be comfortable identifying the <u>'role'</u> a variable is playing.

A variable can be treated as:

1. A constant
2. An independent variable (input)
3. A dependent variable (output), *i.e.* the variable is a function of another variable

Examples:

$$\frac{dx}{dt}, \frac{dy}{dx}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$$

Examples:

a) **One variable functions of *x***: $y = x^2$

$$\frac{dy}{dx} =$$

b) **Related rates problems**: At time *t* a particle is moving along the path $y = x^2$.

$$\frac{dy}{dt} =$$

c) **Implicit functions**: $x^2 + y^2 = 1$

$$\frac{dy}{dx} =$$

d) **Multivariable**: $z = x^2 + y^3e^{6y} - 5xy^4$

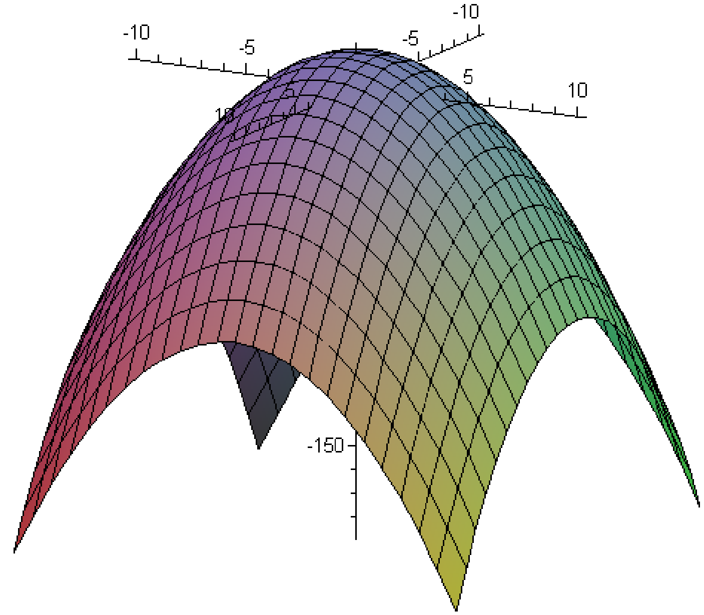
$$\frac{\partial z}{\partial x} = \qquad\qquad , \frac{\partial z}{\partial y} =$$

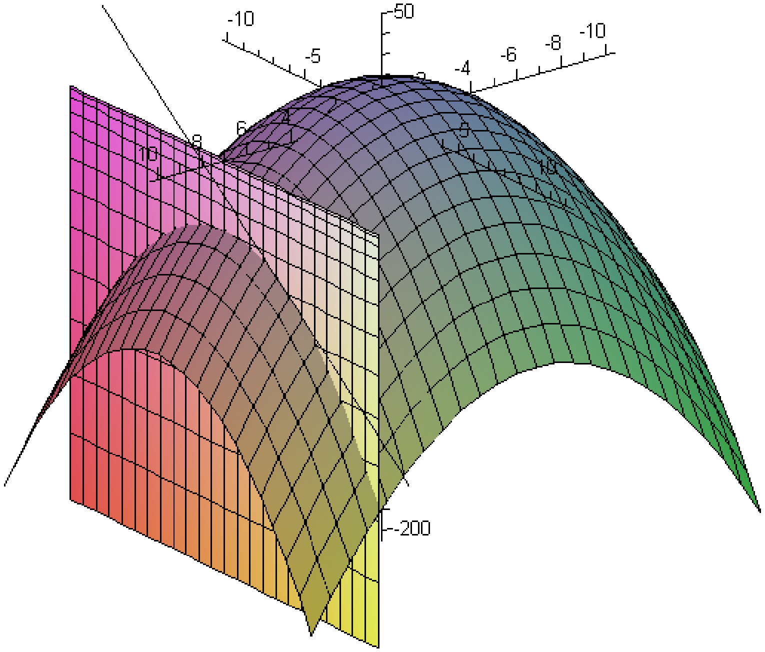
e) **Multivariable Implicit**: $x^2 + y^2 - z^2 = 1$

$$\frac{\partial z}{\partial x} = \qquad\qquad , \frac{\partial z}{\partial y} =$$

Graphical Interpretations:

Pretend you are skiing on the surface
$z = f(x, y) = 15 - x^2 - y^2$

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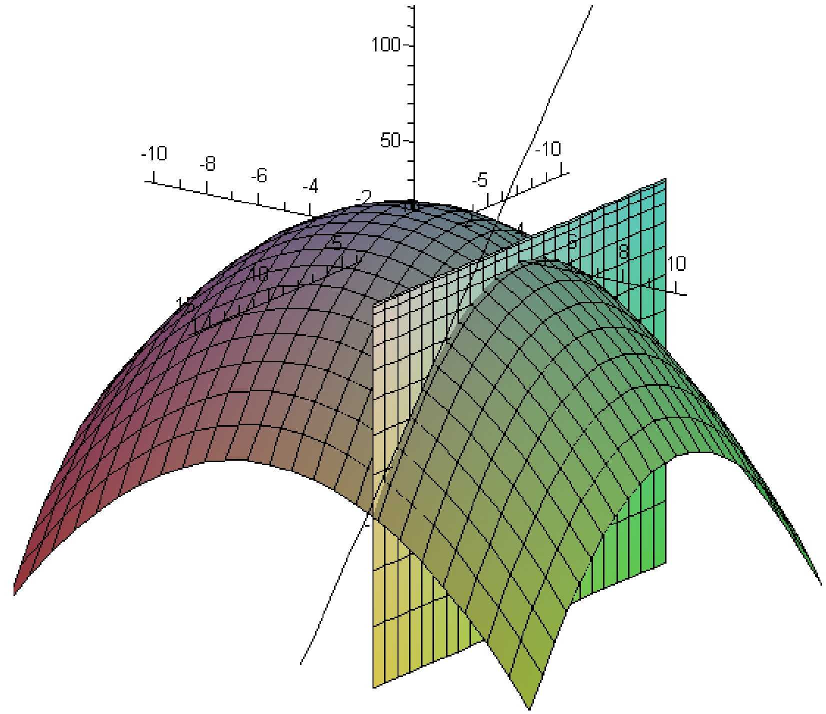
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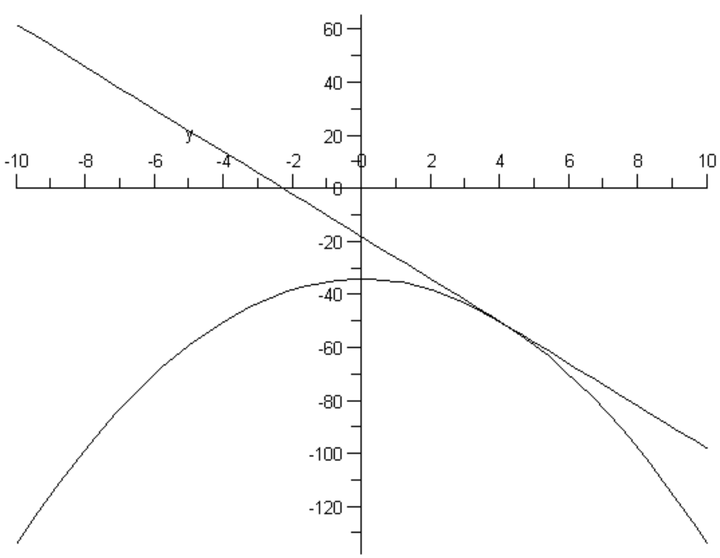
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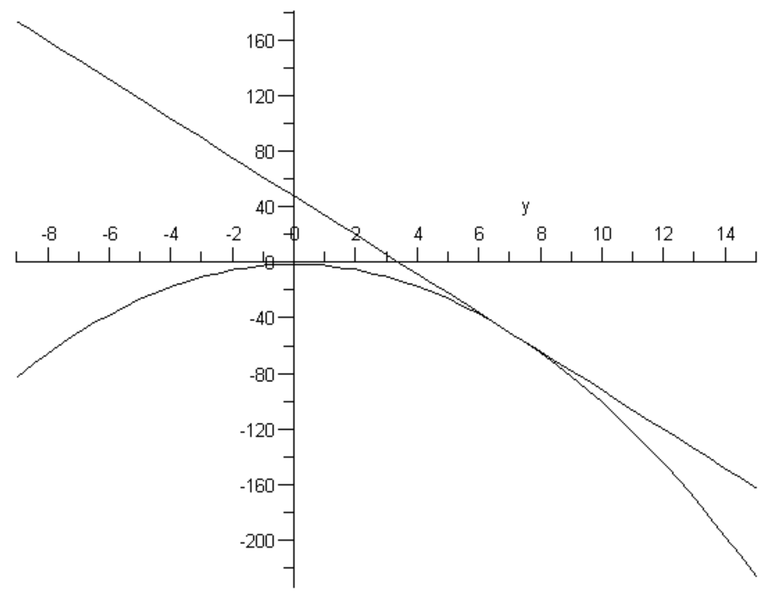
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Second Partial Derivatives

Concavity in *x*-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}(x, y)$$

Concavity in *y*-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}(x, y)$$

Mixed Partials:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}(x, y)$$

Example: Find all second partials for

$z = f(x, y) = x^4 + 3x^2y^3 + y^5$